

## Phase-modulated signals:

(Proakis 101 - 103)

PAM → Amplitude modulation

Phase modulated signal → Phase modulation

$$x_m(t) = \operatorname{Re} \left[ g(t) e^{j \frac{2\pi}{M} (m-1)} e^{j 2\pi f_c t} \right] \quad m=1, 2, \dots, M$$

$\uparrow$

$0 \leq t \leq T$

$M$  possible phases

$$\Rightarrow x_m(t) = g(t) \cos \left[ 2\pi f_c t + \frac{2\pi}{M} (m-1) \right]$$

Obtaining the spanning basis

$$x_m(t) = g(t) \cos \frac{2\pi}{M} (m-1) \cos 2\pi f_c t - g(t) \sin \left( \frac{2\pi}{M} (m-1) \right) \sin 2\pi f_c t$$

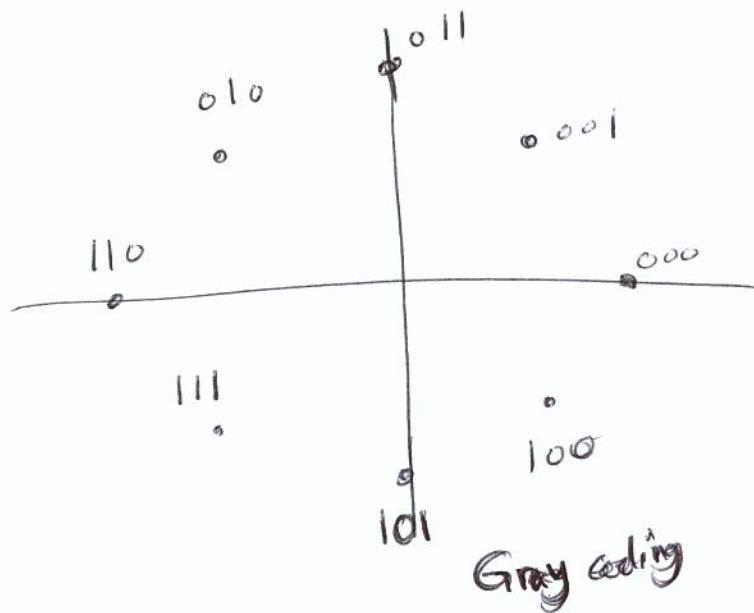
$$\varphi_1'(t) = g(t) \cos 2\pi f_c t$$

$$\Rightarrow \varphi_1(t) = \sqrt{\frac{2}{E_g}} g(t) \cos 2\pi f_c t$$

$$\varphi_2'(t) = -g(t) \sin 2\pi f_c t$$

$$\varphi_2(t) = -\sqrt{\frac{2}{E_g}} g(t) \sin 2\pi f_c t$$

$$M = 2^3 = 8$$



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Arg. energy

$$\|x_m\|^2 = \frac{Eg}{2} \cos^2 \frac{2\pi}{M}(m-1) + \frac{Eg}{2} \sin^2 \frac{2\pi}{M}(m-1)$$

$$= Eg$$

( $\Rightarrow$  what does this say about the detector?)

$$\Rightarrow \text{Arg energy } E_{\text{av}} = Eg$$

Euclidean distance

$$d_{mn} = \|x_m - x_n\|$$

$$= \sqrt{\frac{Eg}{2}} \|$$

$$\left[ \begin{array}{l} \cos \frac{2\pi}{M}(m-1) - \cos \frac{2\pi}{M}(n-1) \\ \sin \frac{2\pi}{M}(m-1) - \sin \frac{2\pi}{M}(n-1) \end{array} \right] \|$$

$\cos(+)\cos(-)$

$\frac{1}{2}$

$\sin(+)\cos(-)$

$$= \sqrt{Eg} \left[ 1 - \cos \frac{2\pi}{M}(m-n) \right]$$

Note that

$$\int_0^T \varphi_1(t) \varphi_2(t) dt = -\frac{1}{\epsilon g} \int_0^T g(t) \sin 2\pi(2f_c)t dt \underset{\approx 0}{\sim}$$

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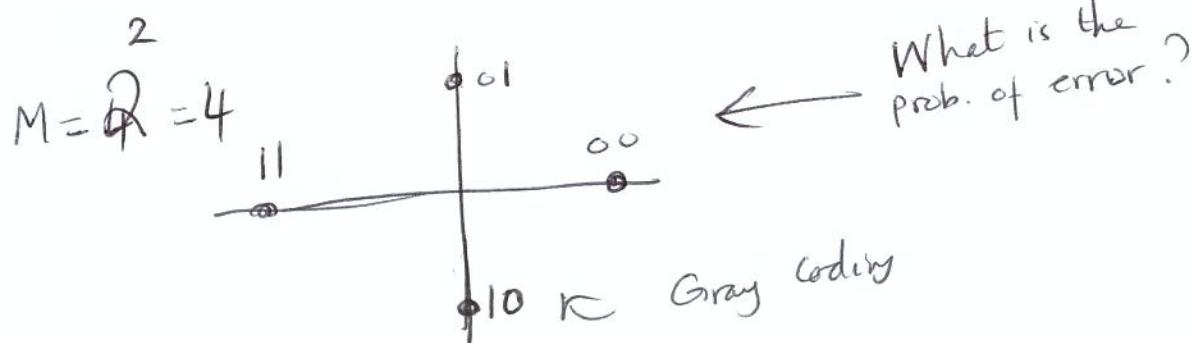
$$\Rightarrow x_m(t) = \sqrt{\frac{\epsilon g}{2}} \cos\left(\frac{2\pi}{M}(m-1)\right) \varphi_1(t) + \sqrt{\frac{\epsilon g}{2}} \sin\left(\frac{2\pi}{M}(m-1)\right) \varphi_2(t)$$

$$\Rightarrow x_m = \begin{bmatrix} \sqrt{\frac{\epsilon g}{2}} \cos\left(\frac{2\pi}{M}(m-1)\right) \\ \sqrt{\frac{\epsilon g}{2}} \sin\left(\frac{2\pi}{M}(m-1)\right) \end{bmatrix}$$

divide  $\frac{2\pi}{M}$  parts  
into  $\frac{2\pi}{M}$  parts  
& put one signal  
here

$$M=2$$
$$x_1 = \begin{bmatrix} \sqrt{\frac{\epsilon g}{2}} \\ 0 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} -\sqrt{\frac{\epsilon g}{2}} \\ 0 \end{bmatrix}$$



$$d_{\min} = \sqrt{\varepsilon g \left(1 - \cos \frac{2\pi}{M}\right)}$$

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Probability of error

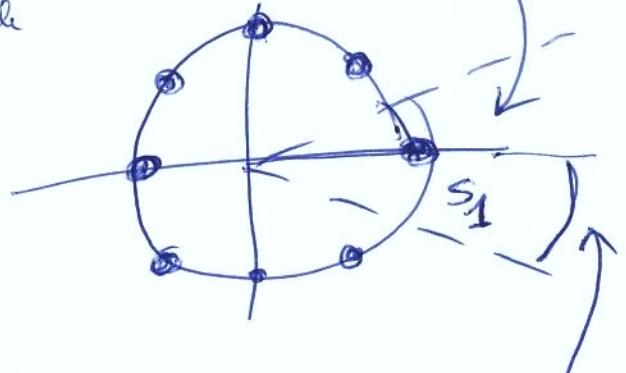
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~~All signals are equiprobable~~

All signals are equiprobable



AWGN

⇒ choose  $s_i$  if

$$\|r - s_i\|^2 \leq \|r - s_j\|^2 \quad i \neq j$$

$$-\frac{\pi}{M}$$

Signals are of equal energy ⇒

choose  $s_i$  if

$$r \cdot s_i > r \cdot s_j$$

$$\Leftrightarrow |r|/|s_i|/\cos\theta_i > |r|/|s_j|/\cos\theta_j$$

$$\cos\theta_i > \theta_j$$

(=)

$$\theta_i < \theta_j$$

(=) angle bet.  $r$  &

Since all signals are equiprobable

$$P_e = P_e/s_1 \text{ transmitted}$$

$$= \cancel{P_e} \cdot 1 - P_e/s_1$$

$$= 1 - P \left\{ -\frac{\pi}{M} \leq \theta_1 \leq \frac{\pi}{M} \right\}$$

~~So~~ we need to find the prob. that  
 the angle of  $r$  is in  $[-\frac{\pi}{M}, \frac{\pi}{M}]$  given  
 that  $s_1$  is transmitted.

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$$r = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{\epsilon g}{2}} \\ 0 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

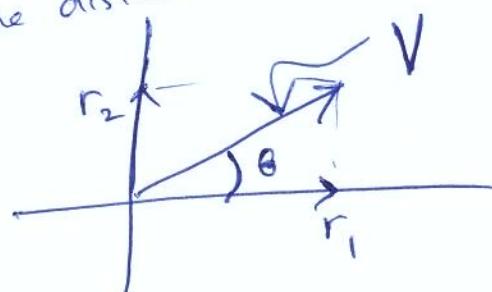
$r_1$  &  $r_2$  are independent Gaussian r.v.'s

$$r_1 \sim N\left(\sqrt{\frac{\epsilon g}{2}}, \frac{N_0}{2}\right)$$

$$r_2 \sim N\left(0, \frac{N_0}{2}\right)$$

$$\Rightarrow p(r_1, r_2) = p(r_1)p(r_2) = \frac{1}{\left(\sqrt{2\pi\frac{N_0}{2}}\right)^2} e^{-\frac{(r_1 - \sqrt{\frac{\epsilon g}{2}})^2}{N_0}} e^{-\frac{r_2^2}{N_0}}$$

Want to find the distribution of  $\theta$  of  $r$



To deal with this prob, we need to do a change of variables

$$f_{v,\theta}(v, \theta) = f_{r_1, r_2}(r_1, r_2) \begin{vmatrix} \frac{\partial r_1}{\partial v} & \frac{\partial r_1}{\partial \theta} \\ \frac{\partial r_2}{\partial v} & \frac{\partial r_2}{\partial \theta} \end{vmatrix} \quad (7)$$

$$\begin{aligned} r_1 &= \sqrt{v \cos \theta} & \Rightarrow \frac{\partial r_1}{\partial v} &= \cos \theta & \frac{\partial r_1}{\partial \theta} &= -v \sin \theta \\ r_2 &= \sqrt{v \sin \theta} & \frac{\partial r_2}{\partial v} &= \sin \theta & \frac{\partial r_2}{\partial \theta} &= \sqrt{v \cos \theta} \end{aligned}$$

$$f_{v,\theta} = f_{r_1, r_2}(\sqrt{v \cos \theta}, \sqrt{v \sin \theta}) \begin{vmatrix} \cos \theta & -v \sin \theta \\ \sin \theta & \sqrt{v \cos \theta} \end{vmatrix}$$

$$= \sqrt{v} f_{r_1, r_2}(\sqrt{v \cos \theta}, \sqrt{v \sin \theta})$$

We finally arrive at

$$P_{V,\theta}(v, \theta) = \frac{V}{\pi N_0} \exp\left(-\frac{1}{N_0}(V^2 + \text{E}_{\text{avg}} - 2\sqrt{\text{E}_{\text{avg}}} \sqrt{V \cos \theta})\right) \quad (1)$$

We need the pdf of  $\theta$ , so we marginalize (integrate) over  $V$

$$\begin{aligned} P_\theta(\theta) &= \int_0^\infty P_{V,\theta}(v, \theta) dv \\ &= \frac{1}{2\pi} e^{-\frac{1}{2}\gamma r \sin^2 \theta} \int_0^\infty V e^{-\frac{1}{2}(V - \sqrt{2r} \cos \theta)^2} dv \\ &= \frac{1}{2\pi} e^{-\frac{1}{2}\gamma r \sin^2 \theta} \int_0^\infty (V - \sqrt{2r} \cos \theta) \sqrt{-\frac{1}{2}(V - \sqrt{2r} \cos \theta)^2} + \frac{1}{2\pi} e^{-\frac{1}{2}\gamma r \sin^2 \theta} \sqrt{2r} \cos \theta \int_0^\infty e^{-\frac{1}{2}(V - \sqrt{2r} \cos \theta)^2} \\ &= \frac{1}{2\pi} e^{-\frac{1}{2}\gamma r \sin^2 \theta} + \frac{1}{2\pi} e^{-\frac{1}{2}\gamma r \sin^2 \theta} \sqrt{2r} \cos \theta \left(0 - \sqrt{2r} \cos \theta\right) \cancel{\int_0^\infty} \\ &= \frac{1}{2\pi} e^{-\frac{1}{2}\gamma r \sin^2 \theta} \left(2 + \sqrt{4\gamma r} \cos \theta \left(0\left(\sqrt{4r} \cos \theta\right) + 1\right)\right) \end{aligned}$$

$$\gamma = \frac{\text{E}_{\text{avg}}}{N_0} \quad (\text{SNR})$$

which is obtained by completing the squares

For high SNR, we can approximate  $P_\theta(\theta)$  as

$$\frac{P}{N_0} \gg 1$$

$$P_\theta(\theta) = \sqrt{\frac{2\gamma r}{\pi}} \cos \theta e^{-\frac{1}{2}\gamma r \sin^2 \theta}$$

$$= 2Q\left(\sqrt{2r} \sin \frac{\pi}{M}\right)$$

$$P_e = 1 - \int_0^{\pi/M} \sqrt{\frac{2\gamma r}{\pi}} \cos \theta e^{-\frac{1}{2}\gamma r \sin^2 \theta} d\theta$$

Use change of variables  $t = \sqrt{2r} \sin \theta$  to get  $e^{-\frac{1}{2}t^2} dt = \frac{2}{\sqrt{\pi t}} \int_0^\infty e^{-t^2} \sqrt{2r} \sin \frac{\pi}{M}$

$$P_e = 1 - \frac{1}{\sqrt{\pi r}} \int_0^{\sqrt{2r} \sin \frac{\pi}{M}} e^{-\frac{1}{2}t^2} dt$$

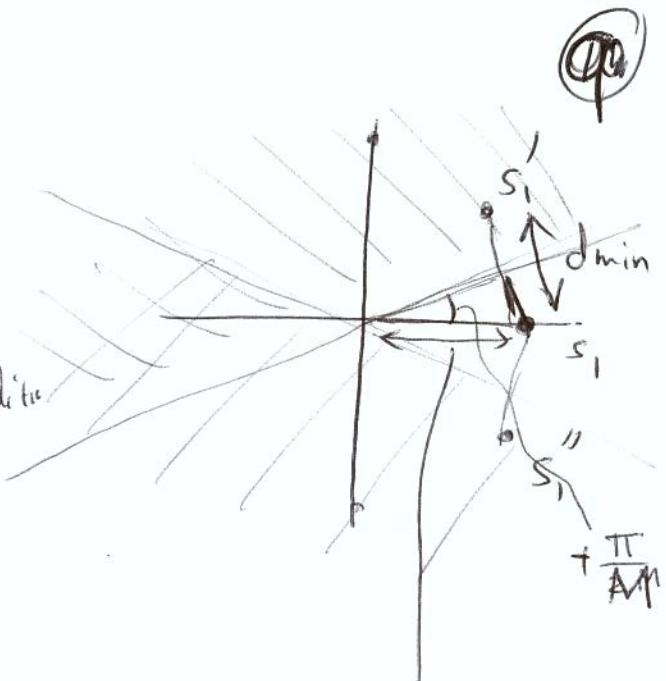
Intelligent union bound approximation

$$P_e = P_{e|S_1}$$

$$\leq P_{S_1 \rightarrow S_1'} + P_{S_1 \rightarrow S_1''}$$

pairwise probabilities

$$= 2 Q\left(\sqrt{\frac{d_{min}^2}{2N_0}}\right)$$



$$d_{min} = 2 \sqrt{E_{avg} \sin \frac{\pi}{M}}$$

$$\text{So } P_e \leq 2 Q\left(\sqrt{\frac{E_{avg}}{2N_0}} \sin \frac{\pi}{M}\right)$$

$$= 2 Q\left(\sqrt{2r} \sin(\pi/M)\right)$$

coincides with previous result.

We can also deduce the lower bound

$$P_e = P_{e|S_1} \geq P_{S_1 \rightarrow S_1'}$$

$$= Q\left(\sqrt{\frac{d_{min}^2}{2N_0}}\right)$$

$$\Rightarrow P_{eL} \geq Q\left(\sqrt{2r} \sin \frac{\pi}{M}\right)$$

Treatment assumes that demod. has perfect knowledge of carrier phase

$$x_i(t) = g(t) \cos(2\pi f t + \frac{2\pi}{M} (\text{mod}) + \phi)$$

Phase is not known

phase could be time variant

reason: phase extracted through a nonlin. process that introduces sign ambiguity.

Soln: Encode in  $f_0$  in phase differences

$$\theta_k = \theta_{k-1} + 180^\circ$$

PSK

1: shift carrier phase by  $180^\circ$  relative to previous phase

$$\theta_k = \theta_{k-1} + 0^\circ$$

0

zero shift relative to previous phase

4PSK

00 :  $0^\circ$  shift

$$\theta_k = \theta_{k-1} + 0^\circ \quad \text{diff} = 0$$

01 :  $90^\circ$  shift

$$\theta_k = \theta_{k-1} + 90^\circ \quad \text{diff} = 90^\circ$$

11 :  $180^\circ$  shift

$$\theta_k = \theta_{k-1} + 180^\circ$$

10 :  $270^\circ$  or  $-90^\circ$  shift.

$$\theta_k = \theta_{k-1} - 90^\circ$$

At Receiver, extract phase at consecutive instants  
phase ambiguity

$$\theta_{k-1} + \phi$$

$$\theta_k + \phi$$

$$\text{diff} = \theta_k - \theta_{k-1}$$

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Rice to pay:

prob. of error is higher than absolute encoding

An error in demod. phase of sig. in any interval

will result in decoding errors over two consecutive signal intervals.

For this reason

$$P_e \text{ differential} \approx 2 P_e \text{ absolute}$$